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Manifolds and Group actions

Homework 12

Mandatory Exercise 1. (10 Points) In this exercise we prove the following

Theorem 1. *Let G be a Lie group and H a closed subgroup. Then G/H is a manifold and its tangent space at eH is $\mathfrak{g}/\mathfrak{h}$. The quotient map $G \to G/H$ is a principal H-bundle.*

a) Check that the action of H on G is proper. Show that G/H is Hausdorff.

For every coset of H in G, we will construct an neighborhood of gH that is equivariantly diffeomorphic to $U \times H$, where $U \subset \mathbb{R}^k$ is open.

b) Let $N \subset \mathfrak{g}$ be a linear subspace such that $\mathfrak{g} = N \oplus \mathfrak{h}$. Define $\Psi : N \times H \to G$ by $\Psi(X, g) = \exp(X)g$. Note that Ψ is H-equivariant from the right, i.e. $\Psi(X, gh) = \Psi(X, g)h$. Check that the differential
$$d\Psi\big|_{(0,e)} : N \oplus \mathfrak{h} \to \mathfrak{g}$$
is the identity. Show that $d\Psi\big|_{(0,g)}$ is bijective for all $g \in H$.

c) Construct a neighborhood V of 0 in N such that $d\Psi\big|_{(X,g)}$ is a bijection for all $X \in V$ and $g \in H$.

d) Show that $\Psi : V \times H \to G$ is a local diffeomorphism.

e) Show that one can find a neighborhood $U \subset V$ of 0 such that $\Psi\big|_{U\times V}$ is an equivariant diffeomorphism to its image (which contains H).

f) Show that every coset of H has a neighborhood of this form.

Mandatory Exercise 2. (5 Points) Let $S^2 \subset \mathbb{C} \times \mathbb{R}$ and $S^1 \subset \mathbb{C}$. Then S^1 acts on S^2 via the formula $z \cdot (w, h) = (zw, h)$.

a) Describe the action (i.e. say if the action is free, proper, transivite, effective; find orbits and stabilizers), and the quotient space S^2/S^1.

b) Show that the antipodal map commutes with the action described above. Explain how this defines an action of S^1 on $\mathbb{RP}^2$.

c) Investigate this action of S^1 on $\mathbb{RP}^2$ and describe the quotient space.

d) Now view $\mathbb{RP}^2$ as the set of lines in $\mathbb{R}^3$, with homogeneous coordinates $[x, y, z]$. Let $SO(2) = S^1$ act on this via $A \cdot [x, y, z] = [A(x, y), z]$. Show that this is the same action as defined in b).

Mandatory Exercise 3. (5 Points) Let $\pi : P \to M$ be a principal G-bundle. Assume that P has a section, i.e a map $\sigma : M \to P$ such that $\pi \circ \sigma(x) = x$ for all $x \in M$. Show that P is isomorphic to $M \times G$ as a principal G-bundle.

Hand in on 17th of July in the pigeonhole on the third floor.